A GRAPHICAL METHOD

FOR

SWING-BRIDGES.

A RATIONAL AND EASY GRAPHICAL ANALYSIS
OF THE STRESSES IN ORDINARY
SWING-BRIDGES.

WITH AN INTRODUCTION ON

THE GENERAL THEORY OF GRAPHICAL
STATICS.

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NEW YORK:

D. VAN NOSTRAND COMPANY,

23 MURRAY AND 27 WARREN STREET.

1892.

PREFACE.

In the analysis of swing-bridges given in the following pages, no acknowledgment of indebtedness, or other reference, to any work which has appeared upon the subject, is therein made, as no book has been consulted in its preparation, although much of the ground has been previously well explored; except that equation (1), of Case III, was taken without material change from Professor W. H. Burr's able work on the "Stresses in Bridge and Roof Trusses." Otherwise the solution of continuity herein given is entirely the result of independent investigation.

In the Introduction no originality

whatever has been attempted. The principles given are well known and established. The texts of other writers have been liberally drawn upon, where principles were found so well expressed as to warrant such quotation; and due credit has been given in each case.

It may be noticed that among the fundamental principles stated, no mention is made of the "Parallelogram of Forces." This term is thought to be inelegant in both its logic and its application. The Triangle of Forces concisely covers the same ground, without the use of superfluous lines.

The article on the analysis of the stresses in swing-bridges, of which the greater portion of these pages consists, was originally intended for publication in some engineering journal, and for such purpose to be complete in itself; which fact explains why it contains matter which otherwise might have been more consistently embodied in the Introduction, which was not thought of until later.

In the hope that this humble effort may contribute something of value to the constantly increasing knowledge of graphical methods of analysis, it is submitted to the perusal of those interested in the subject.

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JACKSON, MICH., 1892.

INTRODUCTION.

The General Theory of Graphical Statics.

Before taking up the analysis of the stresses in swing-bridges, which is the principal feature of this work, the general principles of Graphical Statics, or the graphical investigation of static force, will be briefly noticed, in order that those who are not familiar with the subject may understand the solution as given without being obliged to first study the general theory elsewhere. Those readers to whom the subject of graphical statics is familiar may omit these introductory pages and pass directly to the analysis for swing-bridges.

At the beginning it will be well to

repeat a few well-expressed principles of equivalents in

STATIC FORCE.

"The resultant of two or more forces is a force which singly will produce the same mechanical effect as the forces themselves jointly.

"The original forces are called components.

"In all statical investigations the components may be replaced by their result ant, and *vice versa*.

"The resultant of two unequal forces acting in opposite directions is a single force equal to their difference and acting in the direction of the larger.

"The resultant of any number of forces acting in the same right line is their algebraic sum.

"If three forces are in equilibrium, each will be equal to and opposite in direction to the resultant of the other two.

"If three concurring forces are in equilibrium, and a triangle be formed

by lines drawn in their directions, the sides of the triangle, taken in order, will represent the forces.

"If any number of concurring forces be represented in magnitude and direction by the sides of a polygon, taken in order, they will be in equilibrium."*

This last-expressed principle brings us directly to the subject of

Graphical Statics.

As lines have but one magnitude, length, they may be appropriately taken to represent forces by the distinct properties of magnitude and direction, and the laws of equilibrium of the forces so represented may be determined by applying the principles of Analytical Geometry.

Such geometrical analysis is called Graphical Statics.

It can be applied very successfully, even elegantly, to the investigation of the stresses in framed structures which

* Smith's Mechanics.

are designed to carry specific loads, such as bridge and roof trusses. The subject will here be treated rather toward such end, and not at all exhaustively.

The following principles will be taken as a basis:

1. If two forces which meet at a point be represented, in magnitude and direction, by straight lines drawn end to end consecutively, and respectively parallel to the actions of the forces which they represent, a third line drawn from the beginning of the first line to the unconnected end of the second will represent, in magnitude and direction, the resultant of the two original forces.

2. If the third line be drawn in the same position as before, but in the opposite direction, following around the triangle, it will represent a force in equilibrium with the two first-mentioned forces.

3. When any number of forces act at a point in any direction in the same plane, the force which will be in equilibrium with them can be obtained by drawing

lines which represent the original forces in magnitude and direction, to follow each other consecutively end to end, as the sides of a rectilinear polygon; the remaining side to complete the polygon drawn to the starting-point will represent the force sought.

Note.—This is true in whatever order the forces are taken, but it is usually best to take them in regular order passing in either direction around the point. This practice will generally give the plainest figure and avoid confusion from crossing of lines.

4. If all the forces which meet at a point are known except two and the *directions* of those are known, their magnitudes can be found by first drawing the lines representing the known forces, then completing the polygon by drawing those rspresenting the unknown forces in their proper directions.

This is really the statement of a conclusion rather than a principle; two forces of known directions and unknown magnitudes are determined by substan-

tially the same treatment as one force of which the magnitude and direction are both unknown; each case contains two unknown quantities, which may be determined from a figure having two dimensions, i.e., length and breadth—a plane figure. It is, however, stated as a principle, as it applies with especial value to the determination of the stresses in the members of bridge-trusses and similar structures having a single system of bracing. Each force or stress is considered to act in a straight line between the centres of the connections of each member, and hence its direction is known.

It will be attempted to illustrate the foregoing principles sufficiently to give a clear understanding of them.

Let *a-b* and *b-c* (Fig. 1*a*) represent two forces which meet at a common point; their magnitudes are represented by the respective lengths of the lines, and the direction in which each is exerted is indicated by an arrow. As they cannot be made to take the form of

a closed figure, i.e., as they cannot be so drawn that the end *c* of the line, *b-c*, will be at the starting-point, *a*, they are not in equilibrium. The resultant of their combined force is in the direction *a-c* and of the amount represented by a line drawn between those points; and this resultant, *a-c*, or the combined force of its components, will just be held in equilibrium by a force equal to it in magnitude and of the opposite direction, *c-a*. If the line representing this latter force be combined with the lines *a-b* and *b-c*, they will together form a closed triangle, as *a-b-c* (Fig. 1*d*). Notice that the lines simply represent the forces in magnitude and direction; the point at which they meet is not shown.

The cord *A-B-C* (Fig. 2*a*) is attached and supported at *A* and *C*, and a weight, *W*, is suspended from it at *B*. This weight is a force exerted vertically downward upon the point *B* and, with the forces exerted by the cord in the directions *B-A* and *B-C*, is in equilibrium at this point. The amount of the force *W*

and its direction *B*-*W* are known, and the directions of the forces *B*-*A* and *B*-*C* are known. They must act in the directions of the cords in which they exist and away from the point *B*, as the weight, *W*, causes a tension or "pull" in the cord *W*-*B*, tending to draw the point *B* toward the weight, *W*, which can only be held in equilibrium by tensions in *B*-*A* and *B*-*C*, the amount of which forces may be determined by the application of principles 2 and 4. In what follows it will be noticed that the forces *B*-*A* and *B*-*C* must act in directions away from the point *B* in order to close the triangle.

Draw a line which in magnitude and direction represents the force *W*, as *a*-*c* (Fig. 2*d*), then without removing the pencil draw lines in the directions in which the two unknown forces, *B*-*C* and *B*-*A*, act respectively with reference to the point *B*, to just close at the starting-point, *a*; *c*-*b* and *b*-*a* are such lines and represent the forces sought, and *a*-*b*-*c* is the triangle of the forces which meet at *B*.

The forces in *B-A* and *B-C*, the amounts of which have just been found, are finally resisted at *A* and *C* by being attached to immovable objects at those points. The forces in *B-A* and *B-C* tend to draw the points *A* and *C* toward *B*, and those forces at *A* and *C* which resist this tendency are called the reactions.

Now as, practically, it is not always expedient to support a force or system of forces by inclined reactions, a member which is so designed as to resist compression may be introduced between *A* and *C* which will receive the horizontal* components of the forces in *B-A* and *B-C* so that their vertical components only will be carried by the reactions, and the latter may be designed to simply resist a downward pressure. This compression member is represented by the line *A-C*. The modified figure represents a very simple form of truss.

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* This is not a strictly correct expression, as the components will act along the direction of the line *A-C*.

By drawing the triangle of forces as before, the compressive force in *A-C* may be ascertained. Its direction is known; at *A* it exerts a pressure upon this point in the direction *C-A*. Then at *A* three forces meet: this compressive force in *C-A*, the tensile force in *A-B*, and the vertical reaction which is exerted upward. But the force in *A-B* has been found; it is *a-b* of the triangle of forces (Fig. 2*d*), in which *a-c*, which represents the weight *W*, must also represent the sum of the vertical reactions at *A* and *C*. Now consider the forces acting at *A*: pass over *a-b*, in the triangle of forces, in the direction which the force in *A-B* acts upon *A*, then draw *b-d*, in the direction which the force in *C-A* acts upon this point, to a point from which a line, when drawn vertically upward, will close at *a*; *b-d* will represent the force in *C-A*, and *d-a* the reaction at *A*; *c-d* will represent *W* minus the reaction at *A* (*d-a*), or the reaction at *C*. Thus we have found all the forces acting in and upon the frame

A-B-C. The internal forces which exist in the members of a frame, as *A-B-C*, are called stresses, and the geometrical figure, the lines of which represent the stresses, as *a-b-c-d*, is called the stress diagram.

Fig. 3*a* represents substantially the same truss as Fig. 2*a*, except that in the former the load *W* is transferred to the joint *B* by a member under compression instead of tension.

Fig. 3*d* is the stress diagram for the same; it will be readily understood from its similarity to Fig. 2*d*, which has been explained.

Fig. 4*a* is the same truss inverted. It is the same as the truss shown in Fig. 3*a*, except that those members which in that truss were under a compressive stress, or in compression, are in tension in Fig. 4*a*, and *vice versa*. This figure represents a form of truss very commonly used for short-span bridges; and as this is a practical case, the stress diagram will be followed through in detail. The members of the truss are

designated by figures, which in the diagram of the truss (Fig. 4*a*) are placed in the spaces, and at the angles in the stress diagram (Fig. 4*d*). In the latter the line between any two figures represents the stress in that member of the truss included between the same two characters in the diagram of the truss. The load *W* is situated at midspan, and by the principle of the lever we know that it is received in equal parts by each reaction. We will start with the reaction at *A*, which equals one half the load and is the external force exerted upwards against the truss at this point. Consider, then, the forces acting upon the point *A*: lay off upwards on a vertical line the amount of this reaction, represented to any convenient scale, as 1-2 (Fig. 4*d*); draw 2-3 parallel to that member of the truss (*A*-*b*), and in the direction which its stress (tension) is exerted upon the point *A*, also 3-1 parallel to that member of the truss (*B*-*A*) and in the direction which its stress (compression) acts upon this point, to

close at 1, the starting-point. The tension in *A-b* will be represented to the same scale by 2-3, and 3-1 will represent the compression in *A-B*.

Next consider the forces acting at the point *B*: retrace 1-3 in the opposite direction from which it was drawn, as the compression in *A-B* acts against *B* in the opposite direction from which it acts against *A*; draw 3-4 parallel to that member of the truss and in the direction which its stress (tension) "pulls" upon *B*, and 4-1 parallel to that member of the truss and in the direction which the stress (compression) in it is exerted against this point, to close again at 1. In the stress diagram 3-4 represents the stress in that member of the truss; it is evidently equal to the load *W* and will scale that amount by the scale used in laying off 1-2. The stress represented by 1-4 equals that represented by 1-3. Next, considering the forces acting at *C*, we retrace 1-4 and draw 4-5 and 5-1, which in a sense complete the stress diagram. The line 5-4 equals the line

2-3; this is correct, as they each represent the stress in what is really the same member, *A-C*. The line 5-1 represents the reaction at *C*; it equals the reaction at *A* which at the beginning was found to be one half the load *W* and was represented by the line 1-2; evidently the line 5-2, which represents the sum of the reactions, equals the line 3-4, which represents the load. Each separate triangle in the stress diagram, the sides of which represent the forces acting at some point, is completed, or closed, by a line which returns to the starting-point. One of the many checks to graphics is here given by the fact that it must always be possible to trace each complete polygon, beginning and finishing at the starting-point, without removing the pencil. As many of the lines represent forces which act at two points, i.e., the two ends of a member, and hence belong to two triangles, they must be passed over twice in opposite directions.

This is true of all lines which represent the internal forces (i.e., the stresses

in the members of the truss), when the complete stress diagram is drawn. It is also true of the line which represents the external forces (i.e., loads and reactions) if those forces are parallel, which is generally the case in a truss. The lines representing the resultants of the external forces, whatever may be the directions of the original forces, will always coincide, but that representing the resultant of the loads will be drawn in the opposite direction from that representing the resultant of the reactions.

As has been mentioned, the stress diagram has been in a sense completed; yet for the theoretically complete diagram we have still to consider the joint *b*. The forces in *A*-*C* and *b*-*B* are perpendicular to each other at this point and hence have really no influence upon each other; but it is a point at which forces meet, and it must be taken into consideration in the stress diagram if the latter be in every sense complete.

Now consider the forces which meet at the point *b*: in the stress diagram

pass downward 2-5, the amount of the weight *W*, and in the direction which its force is exerted upon this point, then pass over 5-4, 4-3, and 3-2, the amounts and directions which the forces in those members are exerted upon the same, closing on 2. Each line in the stress diagram has now been passed over twice in opposite directions and the graphical work is complete in every sense. To be thoroughly consistent with good logic the point *b* should have been considered before the point *B*, in order to obtain the stress in the member *B-b*; but as it is evident that this stress is simply a tension equal to the load *W*, and as also it is obtained in the diagram by drawing the polygon (in this case a triangle) for the point *B*, the point *b* was omitted till later. It could have been omitted altogether.

In fact, in this and most ordinary trusses of one span, the two portions between each end and the center are symmetrical and it is only necessary to draw the stress diagram for one half, although by drawing the complete dia-

gram a convenient check is obtained. In this case the only portion of the stress diagram actually necessary to be drawn is the triangle 1-2-3.

Mr. R. H. Bow, one of the pioneers in graphical statics, calls the complete stress diagram the "Reciprocal Diagram of Forces." He states the "chief reciprocal relationship" thus: "Corresponding lines which meet in a point in the one figure form a closed polygon in the other;" and adds: "This is, however, too comprehensive, as it can seldom be altogether complied with." At each angle in the stress diagram meet the lines corresponding to the forces in those which surround or cut off a space in the diagram of the truss, and each space or polygon in the former is surrounded by lines corresponding to the forces in those lines which in the latter meet at a point. This mutual reciprocity is indicated by the positions of the characters in the system of numerals used to designate the lines in Figs. 4*a*

and 4*d*. Corresponding lines are included between like characters.

This system for designating the corresponding lines of the two figures was first used by Mr. Bow, and is described in his "Economics of Construction" in the following language:

"This plan of lettering consists in assigning a particular letter to each inclosed area or space *in*, and also to each space (enclosed or not) *around* or bounding the truss, and attaching the same letter to the angle or point of concourse of lines which represents the area in the diagram of forces. Any linear part of the truss, or any line of action of an external force applied to it, is to be named from the two letters belonging to the two spaces it separates; and the corresponding line in the reciprocal diagram of forces, which represents the force acting in that part or line, will have its *extremities* defined by the same two letters."

In the diagrams which accompany these pages Mr. Bow's system of nomen-

clature has been used, excepting that Arabic numerals have been used instead of letters.

From this author will also be quoted the following explicit

"DIRECTIONS FOR THE CONSTRUCTION OF DIAGRAMS OF FORCES.

"1. Assign a letter to each enclosed area of the truss, also to each division of the surrounding space as separated by the lines of action of the external forces.

"2. The lines in the diagram are to be drawn parallel to the corresponding lines or parts of the truss figure.

"3. The forces acting in lines radiating from a point in the truss must in the *reciprocal* diagram form a closed polygon.

"4. The sides of any uncrossed space, triangle, or other polygon, around or in the truss, must always be represented by lines radiating from a point in the *reciprocal* diagram, and that point is to be

named by the letter assigned to the space or polygon.

. . . .

"7. All the external forces acting upon a truss or other erection must, taken together, be represented in the reciprocal diagram by the sides of a closed polygon." *

This method of graphical analysis is usually known as the Clerk Maxwell method, after the name of one of its earliest English investigators. The writer, however, prefers to designate it as the DIRECT METHOD. As it deals directly with the action of each force, this name seems to be consistent.

It will be well next to briefly notice that somewhat peculiar figure which has been most appropriately called

THE EQUILIBRIUM POLYGON.

This figure will here be defined as: The graphical representation of the

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* The closed polygon, however, may be a straight line.

design of an ideal frame in which the chords are in such perfect equilibrium under the stresses from a given system of external forces that no web members are required.

This definition is, perhaps, unusual; but the writer believes it to be consistent and finds no objection to it. It includes the closing line and, in case of continuity, may include the pier moment, so as to form a complete polygon. The figure is sometimes called a Jointed Frame. It includes the figure quite commonly known as the Funicular Polygon, and also the inverted figure, often called the Linear Arch or Line of Resistance.

It will not be attempted here to consider the Equilibrium Polygon in a manner at all comprehensive, but merely to notice such special class and features of it as may be necessary to intelligently understand the analysis which follows. That figure only will be considered in which the chords of the load curve represent tension and the external forces are vertical. The polygon will be con-

sidered to be composed of imaginary lines having no weight and exerting no influence, except that of resistance to the external forces.

The external forces which will be here considered will be those due to superimposed weights or loads, whose actions are exerted vertically downward, and those vertical reactions which support the actions of the loads when transferred by means of the internal forces of the frame.

The subject of inclined reactions will be noticed in a general way farther on, though not at any length.

Let *AbcdE* (Fig. 5*a*) represent a cord suspended from *A* and *E* and supporting loads at *b*, *c*, and *d*; this cord being free to change its form will assume a certain definite position according to the relative magnitudes of the loads. It represents the tension members of the frame, which in this case compose the lower chord. In the example the loads have been taken as equal. This was done merely for convenience; it is never

necessary. The line *E-A* represents the compression chord of the frame and is called the closing line, because it is always drawn to the starting point of the load curve and closes the polygon. It is not necessarily nor usually horizontal, though it is so in the present case. This chord receives all the horizontal stress from the tension chord, which has been designated as the load curve, so that the reactions are simply vertical. Vertical ordinates at the points of application of the loads, as *b*, *c*, and *d*, between the load curve and the closing line, will be proportional to the bending moments at the respective points due to the loads; they will, indeed, be definite factors of the bending moments, as will be hereafter shown. A change in the length of the cord will affect the lengths of the ordinates, but not their *relative* lengths. The entire system of loads thus sustained will react vertically at *A* and *E* on the principle of the lever.

The graphical analysis of the stresses

of this frame is easily performed by drawing the stress diagram in the manner previously shown; and this stress diagram will be of peculiar value. Three forces meet at each joint of the frame, or angle of the polygon, and if, to start with, any one force be known, from the principles which have been given, the other two can each time be found by drawing the triangle of the forces. The loads being known, the reactions can be readily ascertained by simply applying the principle of the lever. The reaction at *A* will be taken for a beginning; *R*-1 (Fig. 5*d*) represents this reaction, and its force is exerted upward. It may be drawn to any convenient scale. In this case, as the loads are equal and are taken at equal distances from each other and from the reactions—or, in other words, as the loads and panel lengths are equal—each reaction will equal one half the sum of the loads supported, or two and one half loads.

As this force is exerted upward at *A*, pass upwards in the stress diagram its

amount, R-1, drawn to any convenient scale, then draw 1-0 parallel to the line between the same two characters in the equilibrium polygon and in the direction which the cord "pulls" upon A, then 0-R parallel to 0-R of the equilibrium polygon and in the direction which its force is exerted upon A, to close on R; 1-0 and 0-R in the stress diagram will represent the stresses in the corresponding members of the equilibrium polygon; 1-0 will represent tension and 0-R compression.

Now consider the joint b: pass over 0-1 of the stress diagram in the opposite direction from that in which it was drawn, as the force in that member acts upon the joint b in the opposite direction from which it is exerted upon A, then pass downward on the vertical line 1-2, the amount of the load at b, which acts downward at this point, and draw 2-0 parallel to that member of the frame and in the direction which it "pulls" upon this point, to close on 0; thus the stress in 2-0 or bc is found. It will not be necessary to

here follow the stresses through each member of the polygon in detail, as by the aid of the reference characters they can be readily followed through the entire diagram if desired. The lines representing the stresses for the chord *AbcdE* all radiate from 0 in the stress diagram. The lines in the stress diagram (Fig. 5*d*) represent the stresses in those members of the frame (Fig. 5*a*) situated between the same characters. In the frame the characters are placed in the spaces, and at the angles in the stress diagram, as heretofore explained. This completed stress diagram for the equilibrium polygon has been called the Polygon of Forces, or simply the Force Polygon; 1-4 is usually called the load line and 0 the pole; those designations will be used hereafter. The amount that 0 is horizontally distant from the load line is usually called the pole distance, and is often designated simply as *H*; in this case it equals 0-*R*. If a vertical ordinate of the equilibrium polygon be multiplied by this horizontal pole

distance, the product will be the bending moment at that point where the ordinate is taken. This fact is apparent when we consider the equilibrium polygon to represent a frame or truss which is in perfect equilibrium under the assumed system of loads. The pole distance, H, equals the horizontal component of each chord stress, and the ordinates of the equilibrium polygon are the heights of the truss at each panel point. As the entire frame is in equilibrium simply under chord strains and therefore no web strain can exist, it follows that the horizontal component of the chord strains must be uniform throughout the truss; for it can be changed only by the introduction of web members. In a simple truss the bending moment at any point divided by the effective height of truss at that point, which is the lever of the resistance, will equal the horizontal component of the chord strain; and conversely, if we multiply H, the horizontal component of the chord strains in our ideal truss, by the height of truss (or-

dinate of the equilibrium polygon) at any panel point, the product will be the bending moment at that point due to the assumed loads.

Now instead of following this operation in the order given above we may reverse the process, and, by laying off the loads to some convenient scale on a load line, as 1-4 (Fig 5*d*), assuming any convenient pole distance and drawing the radiating lines 0-1, 0-2, etc., we can readily construct the force polygon wholly independent of the equilibrium polygon; then, by drawing the chords of the load curve for the latter parallel to the corresponding radial lines in the former, beginning at one reaction and taking them in order through to the second reaction, and finally drawing the closing line to the starting-point, the equilibrium polygon is drawn for the assumed loads.

Thus the equilibrium polygon is constructed for any system of loads by first constructing the force polygon; and by scaling each ordinate in the former and multiplying it by H, the bending mo-

ments are obtained at all required points.

It is not necessary to space the loads in the equilibrium polygon with the same scale with which the amounts of the loads are laid off on the load line of the force polygon, though it is well to do so when convenient, to avoid confusion. Always measure the ordinates of the equilibrium polygon with the same scale with which the positions of the loads and reactions were located in the same, and for any measurements in the force polygon use the scale that was used to lay off the loads on the load line.

The bending moments, obtained as indicated above, will be the correct moments for the given loads in the given positions on the span, whatever means may be used to *resist* them. Therefore we may consider the equilibrium polygon to be replaced by a truss of any design, and apply the bending moments obtained to the calculation of the stresses. The design of the truss sometimes apparently modifies the bending moment by chang-

ing the position of its application or by resisting portions of it by different systems of bracing. The actual amount of bending moment depends upon the magnitude and positions of the loads and the relative positions of the reactions. A discussion of this, however, does not properly come within the scope of the present article.

The general principles of the funicular polygon (which are comprehended in the equilibrium polygon) seem to have been understood early in the history of graphical statics; but, so far as the writer is informed, they have not, until a comparatively recent date, been to any considerable extent applied to practical uses. Professor C. E. Greene, of the University of Michigan, has made more extensive practical applications of the principles of the equilibrium polygon in Parts II and III of his valuable graphic method for "Trusses and Arches" than has been made elsewhere to the writer's knowledge. He has therein given to the figure the name "Equilibrium Polygon."

That almost phenomenal investigator, Professor Rankine, has shown, substantially, that the equilibrium and force polygons may be drawn for equilibrated forces acting in *any* direction in the same plane. In other words, that the lines representing the external forces (which are in equilibrium through the medium of the polygon or frame) need not necessarily be vertical, nor even parallel; it being necessary only that they may be made to form a closed polygon.*

He has further shown, in substance, though he did not fully develop the fact, that if, from *any* point in the same plane, radiating lines be drawn to the angles of a polygon, whose sides represent a system of forces, the lines so drawn will be parallel to the sides of a frame which will be equilibrated by the given system

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* It would be well, however, to here observe that the *points of application* of the external forces could be such that, though the lines representing them might form a closed polygon, the forces would not be in equilibrium but would form a *couple.*

of forces. In other words, if lines be drawn in proper order, parallel to the radiating lines above mentioned, and intercepted between the lines of action of the forces, the figure so formed will be an equilibrium polygon, and that formed by the radial lines, together with those of the polygon, which represent the system of (external) forces, will be the force polygon or stress diagram for the same.

These principles are important and comprehensive.

From what has been said we may briefly summarize:

1. Lines drawn to represent static forces which are in equilibrium, or which, through the interposition of a transferring medium, as a frame, may be equilibrated, must form a closed polygon.

2. If these forces meet at a common point, they will be properly and sufficiently represented by the respective sides of the polygon.

3. If they do not meet at a common point, but must be transferred by the interposition of a frame, the equilibrium

polygon may always represent the equivalent of the internal forces of such frame.

4. The force polygon is simply the stress diagram of the forces (external and internal) of the equilibrium polygon.

5. The external forces may act in any direction in the same plane. (It is, indeed, not necessary that they act in the same plane, except that forces acting in the same plane are the only ones here considered.)

In the analysis of swing-bridges which follows, the problem of continuity will be solved by means of the equilibrium polygon; and preparatory to the consideration of that solution, it will be well to briefly notice the case of

The Overhanging Arm, or Cantilever.

The foregoing statements have been made with reference simply to a single independent span. But now suppose that the beam or truss continues as an

overhanging arm across and beyond a reaction as AF (Fig. 6*a*). The force polygon will be the same as before, except that there are two additional loads, and *abcdefG* would be the load curve of the equilibrium polygon, drawn as described in the previous example. The last chord of the load curve would be the dotted line f-G if the beam extended to and received one of its reactions at G, and in that case f-G would represent tension in the chord. But the reactions are at A and E, and there is nothing at f-G to resist the resultant of the forces at f, which would be tension in the direction f-G; therefore it must be resisted by compression from the opposite direction. As resistance by the frame to the imposed loads must be supported by the reactions, draw f-E', in the same line with f-G but in the opposite direction, to intersect the vertical through the point of application of the reaction, as at E'. The ordinate E'-e, when multiplied by the pole distance, H, of the force polygon (= 0-3, Fig. 6*d*), will be the bending

moment in the frame at the point of reaction, *E*, due to the load on the overhanging arm at *F*, Fig. 6*a*, and corresponds closely with what is usually called the pier moment in a continuous girder; *f*-*E'*, though really a portion of the load curve, may be considered to be a portion of the closing line, the remainder of which is drawn straight from *E'* to *a*. The closing line must be straight between reactions; it can make an angle only at a reaction, i.e., the application of a force. Where the closing line crosses the load curve, as *x*, is the point of contraflexure, or change of flexure. Where the former lies above the latter, compression will exist in the upper chord and tension in the lower chord, and *vice versa*.

The load at *E* is supported directly by the reaction and does not affect the bending moments of the frame; it may be omitted or not, as desired. Whether or not it be omitted will not affect any ordinate; it may be simply omitted from the load line, Fig. 6*d*; and if there be no load considered at *E*, there will be no an-

gle in the load curve at *e*, Fig. 6*a*. In the figures drawn, however, the load at *E* is included in the load line and the corresponding angle made in the load curve at *e*.

A RATIONAL AND EASY GRAPHICAL ANALYSIS OF THE STRESSES IN ORDINARY SWING-BRIDGES.

THE theoretically correct solution of the stresses in swing-bridges is usually tedious. Sometimes the labor is shortened by the use of approximate methods. Graphic methods receive much and continually increasing favor. The following analysis, composed to a great extent of portions of well-known methods, has the merit of simplicity and is believed to be correct in theory. It necessarily contains much that is not new to those familiar with the subject, and novelty is not largely claimed; yet the writer believes that considerable will be found here that is new, at least in form and manner of application. The tabulation

of the coefficient for the pier moment, the direct graphical use of the latter, by which the reactions are obtained graphically, and the combination of two distinct methods of graphics in Case III, are believed to be original. The solution herein given is not an approximation and it makes no unusual assumptions. It admits and uses the very common and logical assumption that the point of contraflexure actually occurs at the joint nearest to its theoretical position. The graphical correction of the point of contraflexure is thought to be new and worthy of consideration. It does not attempt to go back of a commonly accepted formula for the pier moment, to either question or uphold it, nor even to demonstrate it, but accepts and makes use of such a formula in a simple and practical manner. If an equally simple solution of the stresses for swing-bridges under the condition of continuity has ever been made public the writer has never been so fortunate as to see it.

In the following, the bridge will be

treated as a continuous girder of two spans. Under certain conditions of loading this will give a negative reaction at either free end; but as this can be easily provided against by latching down or raising the ends, it is probably not so great a disadvantage as a negative reaction on either side of the pier, which would often exist if the bridge were designed as a continuous girder of three spans. Nor is the writer fully satisfied that the latter design will never in actual practice give a negative reaction at either abutment. In order that the stresses may be computed for a girder of two spans it is not necessary that the turn-table be center-bearing; with a rim-bearing table the truss may be so designed that it may be treated as having but two continuous spans.

Three methods of accomplishing this are shown in Figs. 3, 4, and 5 (Figs. 1 and 2 apply to center-bearing turn-tables). The arrangement of the center of the truss shown in Fig. 5 is sometimes treated as a case of partial con-

tinuity, considering a portion of the loads to be balanced and a portion unbalanced. It will here be treated simply as a continuous girder of two spans. The writer favors this design, because it is simple to analyze, economical to construct, and gives the shortest possible spans. It offers no ambiguity of stress near the center, which is a great advantage. The pier reactions, though not so uniform as given by the design shown in Fig. 3, are much more uniform than in the continuous girder of three spans. In the design of the center panel shown in Fig. 5, the pier reaction for either span must be upon the adjacent side of the pier, and cannot travel across the pier to the farther side. So far as the stresses in the arms of the truss are concerned, it may be considered as if the center panel did not exist, the truss being shortened the amount of this panel, and the two posts at the pier, except for their own stresses, identified as one post at the center. In practice light counters should be used in the

center panel to prevent excessive tipping while the bridge is swinging, but these counters should be sufficiently loose to take no stress when the bridge is closed, and hence are not to be considered in the calculations.

In the following analysis the reader is supposed to have at least an elementary knowledge of graphical statics, but in explaining it the writer will endeavor to be sufficiently explicit, that such elementary knowledge will be sufficient to fully understand what is here given. To illustrate the method, a bridge will be assumed having 12 panels 20 feet long each, and a center panel of any convenient length and of the design shown in Fig. 5; the bottom chord will be horizontal and the top chord inclined and straight over spans; there will be no counter-braces, but the counter-strains near the ends will be carried by the main web members under a reversion of stress, which will make the analysis rather simpler than though counter-braces were used; the height of truss

will be taken as 20 feet at the hip vertical, and at the pier posts 25 feet. A uniform live load of 2000 lbs. and dead load of 800 lbs. per lineal foot of each span will be assumed for the calculation of the stresses; this will give, for each truss, panel loads of 20,000 lbs. live load, and dead load 8000 lbs.

CASE I. *The Dead Load.*—For the sake of simplicity the dead load will be considered to be concentrated at the lower joints or panel points, though it could as easily be considered divided between the upper and lower joints; the latter is the more proper method of distributing the dead load. One half panel load only of the floor and truss are borne at the outer end, but as a locking or lifting apparatus will materially increase the dead load at this point, it is well to take a full panel load here for the calculations; in the following analysis a full panel load is taken at the end. No attempt will be made to give results, but merely to exhibit the method.

Fig. 6 is an outline of the left one-

half of the truss, and Fig. 7 is the stress diagram for the dead-load stresses. The entire truss is shown to a smaller scale in Fig. 5; both arms are alike. The entire dead load may be considered to be always borne by the pier, unless the ends of the truss are lifted or lowered mechanically.

To aid in following the graphic analysis, the members of the truss and their corresponding stresses have been designated by figures, which in the skeleton of the truss are placed in the spaces, and at the angles in the stress diagram; so that in the latter the line between any two figures represents the stress in the member between the same characters in the former. In the skeleton of the truss, those lines which represent members which under this condition are in tension, and the lines representing the corresponding stresses in the stress diagram, are light lines, while heavy lines represent those members which are in compression, and their corresponding stresses. It is thought that the graphi-

cal figures will thus be very easily understood. In the diagram of the truss, Fig. 6, the members are also designated by letters at the angles or joints; this designation will be used farther on.

For this portion of the analysis is used what the writer prefers to call the direct method of graphical statics, as, by means of the polygon of forces, it gives the stress in each member directly, without further calculation. This solution of the dead-load stresses, and indeed portions of the entire analysis, will be familiar to some readers, but in order that the analysis be complete the whole will be given, though at the risk of being somewhat tedious.

Remembering, then, that the entire dead load is carried by the pier, draw a vertical line, 1–17, Fig. 7, and with some convenient scale lay off downward upon it each panel load for the left arm, beginning with the panel load at *a*.

This line, usually called the load line, represents the sum of the dead loads on one side of the pier. Now consider the

forces acting upon the end joint a : pass downward 1–2 the amount of the end panel load, which is the known external force acting here, and draw 2–3 parallel to that member of the truss and in the direction which its stress acts with reference to this joint, then 3–1 in the direction of its stress with reference to the same to close on 1; 2–3 is the compression in the end panel ab of the lower chord, and 3–1 is the tension in the end post aB. Next take the upper joint B: follow back 1–3, the known tension in aB, and draw 3–4 and 4–1 respectively parallel to those members of the truss and in the directions which their stresses act upon this joint, and the stresses in Bb and BC are obtained. Next take the lower joint b: retrace 4–3 and 3–2 (in the direction opposite to that in which they were drawn, as the tension or compression in any member always acts upon the joint at one end in the opposite direction from which it acts upon the joint at its opposite end), pass downward 2–5 equal to the one panel dead load at b, and

draw 5–6 and 6–4 parallel to those members of the truss and in the directions which their respective stresses act upon this joint, closing on 4. Thus are found the stresses in *bc* and *bC*. Next take the upper joint *C* and retrace 1–4 and 4–6 in the directions which the respective stresses in those members act upon this joint, then draw 6–7 and 7–1 respectively parallel to those members of the truss and in the directions of their stresses with reference to this point, and we have the stresses in *Cc* and *CD*.

It is needless to carry this explanation further, as by the aid of the reference-figures the reader can readily follow out the remaining dead-load stresses.

In practice the scale used should never be smaller than twenty tons to the inch, and much larger would be better if not inconvenient; 50,000 lbs. to the inch is the scale used in the figures, and is too small for practice. With a proper scale, results to the nearest hundreds pounds can be attained with very slight error.

In order to correctly draw the stress

diagram, it is only necessary to draw the lines of each polygon respectively parallel to each of the forces which meet at the joint under consideration. It is well to fix this principle thoroughly in mind, for by strictly following it much error will be avoided. A force may be an external force, as a panel load acting downward, or the stress in a member acting parallel to its length. Compression in a member has exactly the same effect upon a joint as tension in a member which extends in the opposite direction. In the stress diagram draw each line in the direction which the stress in the corresponding member tends to move the point under consideration. It is well, though not necessary, to take first those members in which the stresses are known and pass in order around the joint. If the stresses are not unknown in more than two members which meet at a joint, the stresses in those two are determined by completing the polygon by drawing lines in the proper directions.

The stresses obtained as above indi-

cated will be the dead-load stresses for any position of the bridge, open or closed, so long as the points of support remain unchanged.

CASE II. *Live-load Stresses; Independent Span.*—The live-load stresses will next be discussed on the principle of the lever, as for a single independent span. Fig. 8 is again a skeleton of one-half of the truss, substantially the same as Fig. 6; Fig. 9 is the stress diagram for the chord stresses, and Fig. 10 the stress diagram for the counter-stresses. In these diagrams, again, the character of each stress, tension or compression, is indicated by a light or heavy line.

For the present case the right arm will be considered to be carrying no live load, which will allow the left arm to be treated, for live-load stresses, as though it were an independent span. The maximum compression in the top chord and the end post, and tension in the bottom chord and vertical post at the hip, will occur when the right arm is unloaded and the left arm fully loaded. The

maxima counter-strains in the web members near the end will occur when the right arm is unloaded, and the left arm is loaded from the pier to the foot of a tension member, for the stress in that member, and in the compression member next toward the end of the bridge.

We will therefore consider the left arm fully loaded while obtaining the chord stresses, then, beginning at the left end, remove the load, panel by panel, to obtain the maxima counter-stresses.

The members of the truss and their respective stresses will be designated by numerals as before, see Figs. 8, 9, and 10. Begin by laying off upon a vertical load line the end reaction for a full load (in this case equal to 2½ panel loads) from 1 to 2, Fig. 9, its force being exerted upwards upon a; then draw 2–3 parallel to that member of the truss and in the direction which its stress acts with reference to the joint a; then 3–1 parallel to that member and its stress with reference to this joint, closing on 1. This gives the

maximum compression in *aB* and tension in *ab*. Next take the upper joint *B*: retrace 1–3 in the direction which its stress acts with reference to that joint, then draw 3–4 and 4–1 parallel to and in the directions which their respective stresses act upon this point, closing again on 1, giving the maximum tension in *Bb* and compression in *BC*. Next take the lower joint *b*: retrace 4–3 and 3–2 in the direction which their stresses act with reference to *b*, then pass down the load line the amount 2–5, equal to the one-panel live load at *b*, then draw 5–6 and 6–4 parallel respectively to those members of the truss and their stresses at this joint, always closing the polygon.

It will be unnecessary to follow the operations further in detail, as the reader, aided by the reference-figures, will readily understand them. When a line representing a lower-chord stress, as 11–12 or 14–15, falls below the line representing the upper-chord stresses, as 1–16–4–7–10, the stresses in those members will decrease; the stresses in the vertical

members will change from tension to compression and those in the diagonal members from compression to tension. These stresses will not be maxima for the web members, but will be the maxima compressions in the end post and upper chord and tensions in the lower chord and vertical *Bb*, when combined with the dead-load stresses. It will only be necessary to find those chord stresses which exceed the dead-load stresses in the same members.

To obtain the maxima end counter-stresses requires substantially the same treatment as to find the main web stresses in an independent span. For this it is not an uncommon practice to follow through the diagram the same as for the chord stresses except that 2–5, 5–8, etc., are each made to equal respectively the amount of each panel load which reacts upon the left abutment—in this case $\frac{5}{6}$, $\frac{4}{6}$, etc., of a panel load; thus 4–6″, Fig. 9, would be taken as the maximum stress in *bC*. This practice is illogical, but when the chords are

horizontal it will give correct results. When either chord is inclined, however, the results thus obtained will not be correct, and the error, though small, will in all ordinary cases be on the side of danger. It may be well here to notice further this error which the writer has known to have been habitually made by men in constant practice.

In obtaining the web stress 4–6″ greater chord stresses 2′–6″ and 1–4 will be obtained for *bc* and *BC*, respectively, than can ever occur in those members with this condition of loading, i.e. with no load on *b*; 2′–6″ will represent a greater stress than will ever occur in *bc* from the assumed loads. Now *BC* is inclined and therefore takes a portion of the vertical shear which would otherwise be taken by *bC*, and if, in the stress diagram, the stress in *BC* be taken more than actually exists in that member, then the amount of shear represented to be taken up by it will be proportionately greater than that which it really carries, and 4–6″, or the stress in *bC*, will be less

than that which actually exists in that member. The line 4′–6′ in the stress diagram represents the correct stress in *bC*; it is obviously greater than the line 4–6″ owing to the inclination of the line 1–16–4, etc., which represents the stresses in the top chord. To show that 4′–6′ represents the correct stress in *bC* the diagram will be thus far followed out in detail beginning with the reaction. With all panels of the span except *b* loaded the reaction at the left abutment will be 1–2′, Figs. 9 and 10, or the reaction 1–2, Fig. 9, for the span fully loaded, less $\frac{5}{6}$ of one panel load, which latter is the portion 2–2′ of a load at *b* which would react at this abutment if *b* were loaded.

Now, taking the joint *a*, pass upwards over the reaction 1–2′, then over 2′–3′ and 3′–1; then for the joint *B* pass over 1–3′, 3′–4′, and 4′–1; next for the joint *b* pass over 4′–3′ and 3′–2′, and, as there is no live load at *b*, draw 2′–6′ and 6′–4′ to close on 4′. Thus are found the correct maxima counter-stresses in *Bb* and

bC. For the maxima stresses in the two web members next toward the center take the reaction with *b* and *c* unloaded, which is the reaction last used less $\frac{4}{6}$ of a panel load, and go through a similar operation. It is shown in Fig. 10 and will need no explanation. By deducting the amount of reaction exerted by each panel load on this abutment, as it is considered to be removed, the proper reaction for each condition of loading is obtained; and by taking each reaction thus obtained and following through the diagram as shown in Fig. 10 up to and including the member in question, those live-load counter-stresses are obtained which, when combined with the dead-load stresses, give the maxima stresses of their kind. With this system of bracing they need only be found for those web members in which they exceed the dead-load stresses, which act at the same time and in the opposite direction; 7–9 and 9–10, Fig. 10, are found to be less than the dead-load stresses in the same members, Fig. 7. In practice it is well

to provide for a light counter-strain one panel beyond that in which stress is found, as the dead load may be less than estimated to be, and therefore will counteract a less amount of live-load stress.

If separate counter-braces are used, the case will be somewhat different just here, as the stresses in the verticals will be the sums instead of the differences of the dead and live load stresses. The algebraic sum of these live-load stresses, and the dead-load stresses as before found, will be the maxima stresses for this condition of loading. If counter-braces are used, it is simply necessary to find the live-load stresses in those panels in which the shear from the live load exceeds that from the dead load.

CASE III. *Live Load; Continuous Girder.*—For this case, tensions in the top chord, compressions in the bottom chord, and greatest stresses in the main web members, the maxima stresses will occur in the left arm when the right arm is fully loaded and a load enters upon the outer end of the left arm and advances,

panel by panel, until both arms are entirely loaded. The maximum stress will occur in each member of the chords under conditions which will prevail during this advance of the load over the left arm while the right arm is fully loaded, except near the outer end, where the maxima stresses may occur in the chords and end post from the negative reaction resisted by the latched end, when no load is upon the left arm, or from the half-panel load at the extreme end.

The maximum stress will occur in each tension web member, and the compression member which meets it in the upper chord, when the load covers the left arm from the outer end to the foot of the tension member, the right arm being at the time fully loaded.

These conditions of loading must be considered and treated as a continuous girder. To aid in the analysis of the stresses due to these conditions we will first find what is usually known as the pier moment, i.e., the moment at the pier resulting from the deflections due

to the loads on the spans, causing stress in the chords at the pier.

A formula for the pier moment, M, of a continuous girder of two spans is obtained from the theorem of three moments.

It may be written

$$M = - \left\{ \left[\frac{1}{l_1} \overset{1}{\Sigma} W (l_1^2 - z^2) z \right. \right.$$

$$\left. \left. + \frac{1}{l_2} \overset{2}{\Sigma} W (l_2^2 - z^2) z \right] \div 2(l_1 + l_2) \right\}, \quad (1)$$

in which $W =$ the panel load, l_1 and $l_2 =$ the spans of the arms respectively, $z =$ the distance of each load from the outer end of the span on which it rests, i.e., the left end of the left arm and right end of the right arm, and $\overset{1}{\Sigma}$ and $\overset{2}{\Sigma}$ are the signs of summation for the respective spans. This formula is general and applies to all cases where neither chord is curved.

If the spans are equal, $l_1 = l_2 = l$, and the formula becomes

$$M = -\frac{1}{4l^2}\left[\overset{1}{\Sigma} W(l^2 - z^2)z \right.$$

$$\left. + \overset{2}{\Sigma} W(l^2 - z^2)z \right]. \quad (2)$$

Now if M_0 be taken to represent the amount which each panel load contributes to the value of M, then $M = \Sigma M_0$; and if from both terms of equation (2) we eliminate Σ, we can write

$$M_0 = -\frac{W(l^2 - z^2)z}{4l^2}. \quad . \quad (3)$$

Again, if the panels are of equal length and n be taken equal to the number of panels in each arm, p equal to the panel length, and v equal to the number of panel lengths each load is distant from the outer end of the span on which it rests, then $np = l$, and

$vp = z$. By substituting these values in equation (3) we get

$$M_0 = -\frac{W(n^2p^2 - v^2p^2)vp}{4n^2p^2}$$
$$= -\frac{W(n^2 - v^2)vp}{4n^2}. \quad . \quad (4)$$

If we divide equation (4) by Wp we get

$$\frac{M_0}{Wp} = -\frac{(n^2 - v^2)v}{4n^2}, \quad . \quad . \quad . \quad (5)$$

the second term of which may be tabulated for each position of the load for any number of panels, by simply substituting the values of n and v. The tabulations we may then use as coefficients of Wp to give any value of M_0; for, calling this coefficient (equal to the second term of equation (5)) c, equation (4) can be written $M_0 = cWp$.

This coefficient for the value of M_0 for each panel load of a continuous girder of two equal spans of from 4 to 12 panels each, inclusive, having in each span panels of equal lengths, is given in the following table:

Table of Coefficients for M_0.

(Note.—Consider each coefficient preceded by a negative sign.)

No. of Panels in Span.	Load at b. $v = 1$.	Load at c. $v = 2$.	Load at d. $v = 3$.	Load at e. $v = 4$.	Load at f. $v = 5$.	Load at g. $v = 6$.	Load at h. $v = 7$.	Load at i $v = 8$.	Load at j. $v = 9$.	Load at k. $v = 10$.	Load at l. $v=11$.
$n = 12$	.248	.486	.703	8/9	1.033	9/8	1.155	10/9	.984	.764	.439
$n = 11$	.248	.483	.694	.868	.992	1.054	1.041	.942	.744	.434	
$n = 10$	.2475	.480	.6825	.840	.9375	.960	.8925	.720	.4275		
$n = 9$	.247	.475	2/3	.802	.864	5/6	.691	.419			
$n = 8$	.246	.469	.645	.750	.762	.656	.410				
$n = 7$	.245	.459	.612	.673	.612	.398					
$n = 6$	.243	4/9	9/16	5/9	.382						
$n = 5$	.240	.420	.480	.360							
$n = 4$	.234	.375	.328								

The coefficient is usually expressed decimally, and when multiplied by Wp gives, in each case, a value of M_0 in which the error will always be less than $\frac{Wp}{2000}$. The letters *b, c, d,* etc., are used to designate the respective panel points at which the loads may be situated, beginning at *b*, the first from the outer end, as marked on the diagrams Figs. 6, 8, and 11.

The amount M_0 which each panel load contributes to the value of M will be designated respectively as Mb, Mc, Md, etc. By the aid of the table of coefficients this value for each panel load can readily be determined by simply multiplying the panel load by the panel length, and the product by the coresponding coefficient taken from the table; each value equals Wpc, as before stated. Then by adding these values for each panel which is considered to be loaded, the value of the pier moment, M, for any position of the loads is quickly arrived at. This much facilitates the operation, for the

value of M furnishes a ready key to the solution of the continuity, whatever method is used; for having obtained the value of M, the reactions can be easily deduced.

Before proceeding with the graphical solution it will be expedient to find the values Mb, Mc, etc., for each panel of the left arm and tabulate them; then find the sum of these values for the right arm, which will equal the sum of those for the left arm if the panel loads are equal. For convenience call this sum Ms. Next add and tabulate $Ms+Mb$, this sum plus Mc, this latter sum plus Md, this last sum plus Me, etc., through all the panels of the left arm. These quantities will be designated as Msb, Msc, Msd, etc. If the panel loads are equal, the last sum will check equal to $2Ms$. These results will be the different values of M for each position of the load as it enters upon the left end of the bridge and advances, panel by panel, from the left end toward the pier, until the entire arm is loaded; the right arm being at the same time

fully loaded. At the last position of the load it is obvious that both arms will be fully loaded. These will be all the values of *M*, the pier moment, necessary for determining the maxima stresses of the same character as the dead-load stresses.

This is illustrated in the following tabulation of the pier moments for the bridge and loads as assumed. The data assumed for the bridge were given before Case I was taken up. What data are now to be used will be repeated: the number of panels, n, in each arm are 6, the panel length, p, is 20 feet, and the panel live load, W, equals 20,000 lbs. Now from the table of coefficients, where $n = 6$, we have simply to select each coefficient, c, and multiply by Wp to obtain all the values of M_0, as follows:

VALUES OF M_0 WHICH EACH PANEL LOAD CONTRIBUTES TO M.

$v = 1,\ Mb = 20000 \times 20 \times -.243 = -\ 97{,}200$
$v = 2,\ Mc =$ " " $\times -\frac{4}{9} = -\ 177{,}800$
$v = 3,\ Md =$ " " $\times -\frac{9}{16} = -\ 225{,}000$
$v = 4,\ Me =$ " " $\times -\frac{5}{9} = -\ 222{,}200$
$v = 5,\ Mf =$ " " $\times -.382 = -\ 152{,}800$

$Ms =$ sum of above $= -\ 875{,}000$

CORRESPONDING VALUES OF M.
Right arm fully loaded.

$Msb = -\ 972,200$	$Mse = -\ 1,597,200$
$Msc = -\ 1,150,000$	$Msf = -\ 1,750,000$
$Msd = -\ 1,375,000$	$2Ms = -\ 1,750,000$

$2Ms$ checks Msf.

Having obtained the values of the pier moment for each required position of the load we are prepared to proceed to the solution of the stresses, for which, in this case, the equilibrium and force polygons will be used, and combined with the direct method of graphics.

Draw to some convenient scale a skeleton diagram of one arm of the truss, as $aBGg$, Fig. 11, and from the panel points draw vertical lines downward. In Fig. 11, as in Figs. 6 and 8, the scale used was 50 feet to the inch. The members of the truss and their corresponding stresses are not designated by the system of numerals used in the diagrams of Cases I and II, as that nomenclature could not have been systematically applied to the stress diagram as here combined with the force polygon. Neither has it been attempted to indicate the

character of the stresses by the size of the lines.

At any convenient place lay off the panel loads on a vertical load line in the usual manner; it will be convenient to use the lower portion of the vertical through *a*, as 1–6, Fig. 13. It is not necessary to use for the loads the same scale used in drawing the diagram of the truss, though the same scale (50,000 lbs. to the inch) was used in Fig. 13; but care must be taken to not confuse the two scales, if two are used. For the ordinates of the equilibrium polygon use at unity the scale to which the truss was drawn; for the force polygon and direct graphics use the scale with which the loads were laid off. Select some point, as *O*, usually called the pole, horizontally distant a certain amount, by the scale used for loads, from the load line. This pole distance will be designated as *H* and should be an amount easily used in calculation; in the figure it is taken at

80,000 lbs. It is well to place O somewhat above the center of the load line.

Connect O with the upper point, 1, on the load line by the line O–1 and from some point, as A, Fig. 12, on the vertical line below a, draw A–G_a parallel to 1–O. Lay off G_a–M_a downward from G_a, of the value Ms divided by H, and from this point draw the closing line M_a–A, to close on A. The line O–1′ drawn from O parallel to this closing line to intersect the load line produced will cut off 1–1′, the amount of negative reaction at a when the right arm is fully loaded and no load is upon the left arm.

Now consider the joint a: pass downward 1′–1, the negative reaction just found, and draw 1–a' parallel to aB of the truss, then a'–1′ parallel to ba of the truss to close on 1′. Then 1–a' and a'-1′ will be the maxima live-load stresses in aB and ab, respectively, of the same character as the dead-load stresses, in case the end a of the draw be latched down. Take next the forces acting upon the joint B; retrace a'–1 in the direction

which the stress in aB acts upon this joint and draw $1-a''$ and $a''-a'$ parallel to BC and Bb and in the direction which their respective stresses act with reference to B, then $1-a''$ and $a''-a'$ will be the maxima live-load stresses of the same character as the dead-load stresses in BC and Bb respectively, if the ends of the draw be latched down, or if the negative reaction be less than one-half panel live load.

If the end is not latched down and the negative reaction exceeds one-half panel live load, use the latter instead of the former, as the one-half panel live load would be the amount of force exerted downward upon the truss at a under this condition. As $1-1'$ is less than one-half panel live load, therefore, whether the end be latched down or simply supported and carrying its live load, it will be the proper negative reaction to use in this case. If the end of the bridge be latched down and the negative reaction be of sufficient amount so that $a'-a''$ will exceed one-half panel live load, it is appa-

rent that this condition will also give the maxima stresses in bC and Cc of the kind sought. This is not the case in the example under consideration. The figure $1\text{–}1'\text{–}a'\text{–}a''$ has been extended for the joints of three panels beyond those mentioned; the use of this will be shown in the investigation of the chord stresses.

The point b will now be considered to be loaded; 1–2 on the load line represents the load at b. Draw $O\text{–}2$ and draw $B'\text{–}G_b$ parallel to it. $A\text{–}B'\text{–}G_b$ will be called the load curve for this load. Lay off $G_b\text{–}M_b$, of the value $\frac{M}{H}$, downward from G_b and draw the closing line $M_b\text{–}A$. M in this case equals Msb as tabulated; it is always drawn downward, as it is always negative. The closing line $M_b\text{–}A$ crosses the load curve $A\text{–}B'\text{–}G_b$ at x, which is the theoretical Point of Contraflexure; $M_b\text{–}A$ will be called the theoretical closing line. Now as it is plain that one end of a member cannot be in tension and the other portion under compression at the same time, nor one end be

under stress of either character and the opposite end of the same member be at the same time free from stress, it follows that the contraflexure, i.e., the change from tension to compression, or *vice versa*, must actually occur at a joint, probably the joint nearest to the theoretical point of contraflexure, as C', which will change the value of M. We will designate this joint where the change occurs as the Point of Mechanical Contraflexure. If the theoretical point of contraflexure occurs at the middle of a panel, select the joint which will give the greater stresses to the members in question. Should this occur when we are obtaining the chord stresses with the arm fully loaded, or nearly so, choose the joint at the left, which gives the greater value of M, as this will give the greater stresses of the same kind as the dead-load stresses, which we are seeking. The greatest chord stresses of the opposite kind were obtained when considering the truss to be acting on the principle of the

lever as an independent span, as described in Case II.

As the closing line must be straight, draw $A–m_b$ for the mechanical closing line, passing through C'. The line $O–R$ drawn parallel to this line will divide the load 1–2 into $1–R$ and $R–2$, the portions which react at a and g respectively; $R–2$ will also be the reaction at C from that portion of the truss aC which, under this load, may be said to act as an independent span,—the portion of the truss cg acting at the time as the overhanging arm of a cantilever; $R–2$ will therefore be the compression in Cc. As c is the point of contraflexure there will be no stress in bc or CD. Now consider the joint C: pass upwards over $2–R$, which is the amount of stress in Cc, omit CD as its carries no stress, then draw $R–b''$ parallel to BC and the action of its stresses upon this joint, and draw $b''–2$ parallel to Cb and its stress with reference to this joint, to close on 2, and the maximum tension in bC is obtained.

To show more fully that this is the

proper stress for bC, take that portion of the truss $abBC$ which with this load reacts as an independent span upon a and C. Considering the joint a, pass upward R-1, the left reaction, then draw 1-b' and b'-R parallel respectively to ab and Ba; take next the joint B, pass back on R-b' and draw b'-b'' and b''-R parallel to Bb and BC respectively; next, taking the joint b, retrace b''-b' and b'-1, which represent the stresses in bB and ba, and pass down the load line 1-2, the amount of load at b, and we find that a line 2-b'' drawn from 2 parallel to bC and the action of its stress upon this joint just closes at b'', showing that there is no stress in bc, and that 2-b'' is the stress in bC; R-2, as before stated, is the stress in Cc.

Next consider the load advanced until c also is loaded; 2–3 on the load line represents the load at c. Draw 0-3 and C'-D' parallel to it, continuing the latter to G_c; A-B'-C'-G_c will be the load curve for loads at b and c. Lay off G_c-M_c down-

ward from G_0 of the value $\frac{M}{H}$ and draw the closing line M_0-A; M with this load equals *Msc* as tabulated. This time the closing line crosses so near to D' that, without change, it will be considered to pass through D' and to be the correct mechanical closing line. O-R', which coincides with O-2, cuts off the pier reaction R'-3. Now consider the joint D: pass upward 3-R', which is the compression in dD, and, omitting DE, which with this load has no stress, draw R'-c' and c'-3 parallel to CD and Dc respectively; then 3-c' will be the tension in cD.

This again can be more fully proven by drawing the complete diagram of that portion of the truss which is at the left of the point of contraflexure, beginning with R'-1, the left reaction. It is unnecessary to give it here in detail.

Next consider the load advanced until the joint d is also loaded: 3-4 will be the additional load and D'-E' will be drawn parallel to O-4, and produced to

G_d; A-B'-C'-D'-G_d will now be the load curve. G_d-M_d will equal M $(= Msd)$ divided by H. M_d-A will be the theoretical and m_d-A the mechanical closing line, and E' the point of mechanical contraflexure. O-R'', drawn parallel to m_d-A, will cut off R''-4, the pier reaction which is also the compression in Ee. Consider the joint E: draw R''-d' parallel to DE and d'-4 parallel to Ed; d'-4 will be the tension in dE.

Next advance the load to e; 4-5 on the load line will be the additional load; draw O-5 and draw E'-F'-G_e parallel to it; A-B'-C'-D'-E'-G_e is the new load curve. Lay off M at G_e-M_e of the value $\frac{Mse}{H}$; M_e-A will be the theoretical closing line, but m_d-A will again be the mechanical closing line. The last load e will be at the point of mechanical contraflexure. The pier reaction will now equal the stress in Ee plus the load at e, which fact will slightly change the diagram. It is to be borne in mind that any load at, or to the right of, the point

of contraflexure will cause no reaction upon the left abutment. Now, considering the joint e, pass downward the amount of the pier reaction R''-5, which is the sum of the two forces acting downward at this point, then draw 5-e' and e'-R'' parallel to eF and fe respectively, closing on R'', as there is no stress in de; 5-e' will be the stress in eF. Consider the point F: retrace e'-5, draw the line 5-e'' parallel to FG, and e''-e' parallel to fF, closing on e', as there is no stress in EF; e'-e'' will be the compression in the post Ff.

Now advance the load upon f and the entire arm is loaded. The additional load will be represented by 5-6 on the load line and A-B'-C'-D'-E'-F'-G_f will be the complete load curve. The theoretical closing line passes nearly midway between E' and F', and E' is again selected as the point of mechanical contraflexure; for this position will give greater stresses in the members to be considered than if F' were chosen. The pier reaction is 6-R'', and this is ob-

viously the maximum compression in *Gg*. The new pier moment G_f-m_d, multiplied by *H* and divided by the depth *Gg* of the truss, will be the maximum stress in *GG* of the top chord and compression in *ff* of the bottom chord. Now consider the joint *G*: pass upward the pier reaction 6-R'', which is the stress in *Gg*, draw R''-g' parallel to *GG* and equal to the stress in that member, g'-f' parallel to *GF* and its stress with reference to *G*, and f'-6 parallel to *Gf* and its stress with reference to the same, to close on 6; f'-g' is the maximum tension in *FG*, and and 6-f' that in *fG*. Thus are found the maxima live-load stresses in the main web members and in both chords in the middle and next adjacent panels, for a line drawn from f' perpendicular to R''-g' will cut off f''-g', the maximum compression in *ef*.

G_f-m_d scales about 29.8, which multiplied by 80,000, (*H*,) gives a pier moment of 2,384,000 ft.-lbs. for the full load, instead of 1,750,000 ft.-lbs. as calculated and tabulated. This rather

large difference is due to the change from the theoretical to the mechanical position of the point of contraflexure; for G_f-M_f multiplied by H will give the moment as calculated. That the contraflexure, or change of flexure, actually occurs at a joint is probably undoubted. In a case like the present, where the theoretical position of the change of stress occurs about midway between two joints, it is impossible, with our present knowledge of the action of strains, to definitely decide at which point the change really occurs. The writer believes that good judgment will select that joint which will give the greater stresses in the members under consideration.

The remaining maxima live-load stresses of the same nature as the dead-load stresses, i.e., tensions in the upper and compressions in the lower chord, will be easily obtained by scaling the bending moments directly from the ordinates of the equilibrium polygons, where the closing lines falls below the broken lines or

load curves. It could indeed occur that the stresses as just found in *ef* and *FG* would not be the maxima stresses in those members, in which case they also would be taken from the ordinates of the polygon. The maximum ordinate at each panel point may be easily found by inspection. For any loading the value of *M* can be quickly calculated by the aid of the table of coefficients and the closing line to the polygon drawn. A few principles regarding the loads for these stresses may be mentioned.

It is plain that the condition of loading must in each case be such that the point of contraflexure will be somewhere at the left of the member in which the stress is sought, in order that the stress be of the kind sought; the portion of the truss in which the member is situated will, under this condition of loading, act as an overhanging arm, which supports at the point of contraflexure what may be considered as the end of that portion of the truss at the left of this point, which is, with this condition,

acting as the truss of an independent span. Now, evidently, when thus conditioned, a load at the right of the member in question cannot cause stress of the kind sought in that member; but, on the contrary, by changing the point of contraflexure to a position nearer the pier, will always tend to lessen the stress of this kind. Therefore the maxima upper-chord tensions and lower-chord compressions will always be caused by loads to the left of the member under consideration, the right arm being at the time fully loaded. In the panels near the end the maxima stresses of this kind may be caused simply by the negative reaction being resisted by the latched end; or, if the end be not latched, by the half-panel live load at the end, no other portion of the left arm being loaded. In the present case the negative reaction will be the same whether resisted by either condition; it causes maxima chord stresses of this character in the four panels nearest the end, or from a to e and from B to F.

An examination of the equilibrium polygons will always readily show in what panels this condition will obtain, and the amounts of stress can be arrived at, either from the moments by scaling the ordinates of the polygons, or by continuing the diagram 1-1′-a'-a'' by the direct method through the required members, as shown at the top of Fig. 13. The latter is perhaps the easier method. All of the maxima live-load stresses of this character will usually be found under some of the conditions of loading just considered for the main web stresses and for which the polygons are already drawn.

From the equilibrium polygons drawn for conditions of loading as above, we have only to select and scale, at each panel point, the maximum ordinate between the broken line or load curve of a polygon and its closing line, where the latter lies below the former, multiply it by H, and divide by the depth of the truss at the point in question (i.e., at the vertical between the same two diagonals

between which the member under consideration is situated), and we have the maxima compressions in the lower chord and horizontal components of the tensions in the upper chord due to live load.

For the upper-chord stresses draw a line parallel to the lower chord (horizontal), and near it a line parallel to the upper chord. On the former lay off the lower-chord stresses; at the measures, perpendiculars erected to it, cutting the latter, will intercept on it the upper-chord stresses, i.e., those in the members included between the same diagonals. See g'-f'' and g'-f', Fig 13.

These stresses when combined with the dead-load stresses give the maxima stresses of the same nature as the dead-load stresses, and the analysis is completed if the ends of the bridge are neither latched down nor raised.

Ends Latched Down.—If the ends of the bridge be latched down, the negative reaction at the outer end of the right arm, when not loaded, caused by the deflection due to loads upon the left arm

will be resisted, which will cause the bridge to act as a continuous girder, and we cannot consider either arm to act as an independent span under this condition of loading ; this will reduce such maxima stresses as have been found to occur under this condition. Then in that portion of the analysis given in Case II, for the chord stresses we must start with the reaction 1-2 equal to the reaction at *a* with the left arm fully loaded and the right arm unloaded, the bridge considered as a continuous girder. The pier moment will equal *Ms*, and the reaction is readily found. Then for the counterstrains in the web members, as each panel is removed, beginning at the outer end, to obtain the maximum shear we must again each time calculate the reaction and lay it off on the load line instead of 1-5, 1-8 ; etc., or what amounts to the same thing, make 2-5, 5-8, etc., equal to the difference between each new reaction and the one previously used. The reaction at the abutment *a* will each time be equal to the corresponding

reaction of the same system of loads upon an independent span, plus the pier moment for that loading divided by the length of the span of the arm ; or $R + \frac{M}{l}$, in which R equals the reaction for the span, l, considered, on the principle of the lever, as a single independent span. M is easily obtained for any position of load by the aid of the table of coefficients. Remember that M is always negative. These values of the reaction have simply the theoretical points of contraflexure. The reader can in each case correct it graphically to the point of mechanical contraflexure, as previously shown, should he choose to do so. From what follows it will be seen that this correction or a further explanation is not considered necessary.

The writer believes that in obtaining the maxima compressions in the top chord and end post, tensions in the bottom chord and first web member, and counter-stresses in the balance of the web members, it is best always to con-

sider the arm to be acting as an independent span; because the load may at times come upon the bridge when the ends are not latched, causing either arm to act as an independent span, so far as the live-load stresses be concerned; in which case the strains would be greater than those stresses obtained by considering it as one span of a continuous girder. The treatment for the negative reaction at the latched end has been given.

ENDS LIFTED.—In case the ends are lifted by a lifting apparatus, or what may be termed an artificial reaction, applied to the ends, calculate first the dead-load stresses as given in Case I for the bridge when unloaded and swinging, but for those dead-load stresses which are to be combined with the live-load stresses use a positive reaction at the abutment end equal to the estimated force which is to be applied to lift it. This force should exceed any possible negative reaction at this point. The diagram will be somewhat similar to that for Case

II, Fig. 9, and is so simple and so readily understood that an explanation of the method of constructing it will not be given. The known external forces will be this artificial reaction exerted upward at the abutment, and the dead loads acting downward at the panel points. Begin with the artificial reaction.

For important structures it is best to combine both cases of dead-load stresses with the live-load stresses, and select the greatest results, as the live load may at times come upon the bridge when for some reason the ends have not been raised. Combine Case I, as herein given, with Case III; and combine Case I, with the artificial reaction applied to the end, with Case II. These stresses thus combined will be maxima for all conditions of ends raised or ends simply supported.

CONCLUDING REMARKS.—In the entire foregoing discussion it has been assumed that each panel point receives its load wholly independent of any other panel load; this, for uniform loads, is a com-

mon and perhaps usual assumption. For all the ordinary types of bridges in which the floor load of each panel is carried to the floor-beams at the panel points by means of stringers, or any equivalent arrangement, this assumption is not correct; for no individual beam can receive the full load from a stringer: the adjacent beam or abutment must receive a portion of the load. Therefore no panel point can be fully loaded without each adjacent panel point carrying at least a half panel load; from which fact it follows that the maxima shears will actually be somewhat less than found under the usual assumption. The error, however, is small, and is always on the side of safety; and the writer believes that the practice of finding the web stresses according to the usual custom of assuming each load to act independently is a creditable one. The web members may well be allowed this much favor in the calculations.

In the example given, the writer has selected a very simple type of bridge, and

has endeavored to explain the application of the method quite fully, believing that from so doing the analysis will be more readily understood than if more complicated case had been chosen. When the method herein given is thoroughly comprehended, it will be readily adapted to several different types of trusses, though not to all. In the design of the trusses *in the spans* the method applies, with slight modifications, to any single system of triangulation. No difficulty is presented by the use of counterbraces in the panels near the ends, and the treatment for such cases is so nearly the same as that given, and so simple, that it was not deemed necessary to be given. The slight difference occurs in Case II.

Figs. 1 to 5, inclusive, show as many different arrangements of the truss at the pier to which this analysis will apply; it is thought that the practical designer will find it adaptable to a sufficient number of available designs that he will consider it to be unnecessary labor to go through the tedious calculations of a

three-span design. Figs. 1 and 2 show arrangements at center of truss for center-bearing turn-tables. For the arrangement shown in Fig. 1 all panels must be of equal length, which may not always be found convenient. In that shown in Fig. 2 the center panel may be of any convenient length, as the pier reactions of the truss are upon the ends of beams which are independent of it, and for the stresses in the arms of the truss the center panel may be considered not to exist. The greatest strain that can come upon a counter in the center panel will be the strain from one half the shear from the greatest amount of unbalanced loads, which latter will be one half the pier reaction from a full live load on one arm, the other arm being at the same time free from load.

Fig. 3 is susceptible of two distinct arrangements. If arranged as shown in the figure, the center panels may be omitted in the calculations of the stresses in the arms; or, if desired, the top chords may continue straight to the cen-

ter, as in Fig. 1. In the first case the center panels may have any convenient length and the arms considered to react at the adjacent vertical posts; in the latter case all panels must be of equal length and the two arms considered to receive the pier reaction at the center of the top chord. In Figs. 4 and 5 the counters in the center panels are merely inserted to prevent excessive tipping of the bridge when swinging. They should have a small amount of play at their connections so as to take no strain when the bridge is closed, and hence have no place in the calculations of the stresses, for which reason they are not shown with full lines. Fig. 5 represents the entire truss of the type herein treated.

For unequal panel lengths in the arms, the table of coefficients will not apply; for such cases equation (1) may be used to find the pier moment, and the remainder of the treatment will be substantially as given.

It is realized that much has been given,

in describing this analysis, which to some will seem superfluous; the writer begs the indulgence of such, believing that others, not so familiar with graphical methods, will appreciate the rather elementary manner in which the description of the method has been attempted.

The writer believes that for the ordinary types of swing-bridges the foregoing analysis will compare favorably with any that has been placed before the public. In the usual swing-bridge the spans of the arms are equal and the panels in them are of equal length, and for such the table of coefficients for the pier moment furnishes a ready and convenient key to the problem of continuity for any design equivalent to a two-span girder. The table applies for unequal as well as for equal panel loads, and stresses may be calculated for concentrated loads.

In conclusion, the writer would say that he can discern no satisfactory reason why the ordinary case of a swing-bridge should be treated as a continuous girder of three spans. So long as the services

of a bridgeman be required to turn the bridge, there can be no reasonable objection to latching down, or raising, the ends ; and with either of those conditions the continuous girder of two spans is superior to that of three spans ; for it gives more uniform reactions on the two sides of the pier, affords a much simpler treatment of the stresses, and is free from ambiguity of stress in the members near the pier. It is certainly a better policy to so design a bridge that the stresses may be readily and definitely determined than to choose a complicated design involving any degree of uncertainty.

APPENDIX A.

COMPARATIVE REACTIONS UNDER FULL LOAD.

It is a well-established principle that if a continuous girder of two equal spans consists of a solid beam or such other design that its load may be uniformly distributed throughout its length, with both spans fully loaded, the total load on either span will react in the proportion of three-eighths upon the end abutment and five-eighths upon the centre pier. A girder composed of an open system of bracing cannot be uniformly loaded throughout its length, though it may carry uniform concentrations at the panel points, which approximate rather closely to a uniformly distributed load. The reactions from such a girder, continuous over two spans, will only approximate the principle stated above; the approximation will be more or less close according as the concentrations are situated at more or less frequent intervals upon the spans.

From the table of coefficients for M_0 we may quite closely express the ratios of these reactions on spans of from four to twelve intervals, or panels. The following tabulation shows the comparative ratios, expressed to the third decimal place, and is, indirectly, a proof of the reliability of the table of coefficients, and the formula from which it was derived.

Concentrations.	Abutment.	Pier.
Uniform load..............	.375	.625
12 panels..................	.376	.624
11 "	.376	.624
10 "	*.37625	*.62375
9 "	.377	.623
8 "	.377	.623
7 "	.378	.622
6 "	.379	.621
5 "	*.380	*.620
4 "	.383	.617

* Exact decimals.

The above ratios include the half-panels which rest directly upon the abutment and pier, which is necessary in order to afford a consistent comparison with a uniform load. Both spans are considered to be fully loaded.

APPENDIX B.

THE PIER MOMENT FOR UNEQUAL SPANS.

In the preceding analysis for swing-bridges the problem of continuity was solved by means of a table of coefficients from which is readily obtained each value of M_0,—the character M_0 being understood to represent the amount which each loaded panel affects, or contributes to, the pier moment, M, in a continuous girder of two equal spans, having equal panel lengths throughout the spans. It will now be shown that this table of coefficients is applicable to unequal spans with very nearly the same facility with which it can be applied to equal spans. It is not necessary that any dimension of one span shall equal the corresponding dimension in the other span; it is only

necessary that the panel lengths be uniform throughout each individual span.

The following formula for the pier moment, M, of a continuous girder of two spans was given, and from it were tabulated the coefficients for M_0 (see equation (1), Case III, page 63):

$$M = -\left\{\left[\frac{1}{l_1}\overset{1}{\Sigma} W(l_1^2 - z^2)z\right.\right.$$

$$\left.\left. + \frac{1}{l_2}\overset{2}{\Sigma} W(l_2^2 - z^2)z\right] \div 2(l_1 + l_2)\right\}. \qquad (a)$$

In this equation the same symbols have been used and have been given the same significance as in equation (1).

Let M_1 represent the amount which any panel load contributes to the pier moment of a continuous girder of two (equal or unequal) spans, and let l_0 represent always the span on which the load is situated. (z must necessarily refer to the same span.) Then $M = \Sigma M_1$, and, omitting the summation, we may write

$$M_1 = -\left[\frac{1}{l_0} W(l_0^2 - z^2)z \div 2(l_1 + l_2)\right],$$

or, in a different form,

$$M_1(l_1 + l_2) = -\frac{W(l_0^2 - z^2)z}{2l_0}. \quad . \quad (b)$$

Now, let $np = l_0$ and $vp = z$, following the former notation (given in Case III), but referring always to the span on which the load is situated, and equation (b) becomes

$$M_1(l_1 + l_2) = -\frac{W(n^2p^2 - v^2p^2)vp}{2np}$$

$$= -\frac{Wp^2(n^2 - v^2)v}{2n}. \quad . \quad (c)$$

Again, if we divide both terms of equation (c) by $2np$, substituting in the first term its equivalent value, $2l_0$, we have

$$M_1\frac{l_1 + l_2}{2l_0} = -Wp\frac{(n^2 - v^2)v}{4n^2}. \quad (d)$$

The last term of this equation is the same as the expression formerly obtained for the value of M_0, and the expression

$\frac{(n^2 - v^2)v}{4n^2}$ is exactly that which in Case III was tabulated for the coefficients of M_0 and designated as c. Using this designation for the above expression, and remembering that c is always negative, equation (d) may be written

$$M_1 = c\,Wp\frac{2l_0}{l_1 + l_2}. \quad . \quad . \quad (e)$$

From equation (e) it is plain that, to find the value of M_1 for two unequal spans, we perform precisely the same simple operation as for finding the value of M_0 for equal spans, then multiply the result by twice the length of the span on which the load is situated, and divide by the combined length of the two spans.

To express it algebraically,

$$M_1 = M_0\frac{2l_0}{l_1 + l_2}.$$

If $l_1 = l_2$, then will either of those symbols equal l_0, and the expression $\frac{2l_0}{l_1 + l_2}$ will become unity.

When the spans are unequal, it is necessary to calculate the stresses for each span separately, which involves about double the amount of computation necessary for two equal and symmetrical spans. But the former exceeds the latter only in the *amount* of computation required; it is equally simple, as has been shown.

It is not thought that it will be necessary to give an example of the application of the method to unequal spans, as it follows, generally, the method for equal spans, and it is believed that the explanation given will be sufficient; but it may be well to mention that with unequal spans it is necessary to remember that for the values *Msb*, *Msc*, etc., we must combine the values *Mb*, *Mc*, etc., for one span with the value *Ms* of the other span, when considering a moving load on one span with the other span fully loaded.

It may be of interest to notice that a load upon the longer span affects the pier moment much more than an equal

load relatively situated on the shorter span, owing to the greater deflection of the former.

The solution of continuity for two spans, as herein given, though of not quite universal application, is so very nearly general that it may be very successfully applied to the calculations of Ordinary Swing Bridges.

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